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RESEARCH ARTICLE

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Unifying adaptive control with the nonlinear PI methodology: Designs for unknown strict-feedback nonlinear systems with nonsmooth actuator nonlinearities

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Summary

In this paper, we extend the nonlinear PI control methodology within an adaptive control framework. An adaptive nonlinear PI controller is proposed for output tracking of strict-feedback nonlinear systems with nonsmooth actuator nonlinearities and unknown control directions. The current approach relaxes the standard assumption of known bounds for the associated system nonlinearities made in earlier nonlinear PI schemes. New theoretical boundedness results have been proved that enable the successful combination of backstepping and linear parametric approximators with the nonlinear PI approach and ensure semiglobal approximate tracking of the output to some reference trajectory. Following recent extensions of the nonlinear PI method to strict-feedback systems, the intermediate virtual control laws are derived through suitable integral equations. Simulation results are also presented in this paper that verify our theoretical analysis.

KEYWORDS

actuator nonlinearity, adaptive control, backstepping, nonlinear PI

1 | INTRODUCTION

The nonlinear PI method was originally proposed by Ortega et al[1] as an alternative approach to parameter adaptation. The central idea of this technique revolves around the general immersion and invariance methodology.[2] Specifically, a stable error equation is defined with a perturbation term that has at least one root, and the nonlinear PI gains are designed so that trajectories converge toward the root.[1,3]

In certain control applications, there are cases of systems for which the control directions may be unknown. These include, for example, the uncalibrated visual servoing problem[4] and autopilot design of ships.[5] Moreover, actuation sign errors have been observed in attitude control of microsatellites.[6] The unknown control direction problem has been mainly addressed with the use of the so-called Nussbaum gain methodology.[7] This approach has received increased popularity over the years as it provides a design tool for general classes of systems and can be combined successfully with parameter adaptation algorithms.[8-23]

On the other hand, the nonlinear PI method provides an effective alternative strategy to the unknown control direction problem (see section 6 in the work of Ortega et al[1]). Compared with Nussbaum gains, it was shown that the nonlinear PI technique is more robust to certain classes of unmodeled dynamic perturbations (for details, see example 8 in the work of Georgiou and Smith[24] and section 6.1 in the work of Ortega et al[1] and in our other works[25,26]). Extensions to regulation of strict-feedback nonlinear systems have been considered recently in our other work.[27] Applications to consensus problems for systems with unknown high-frequency gains and switching topologies have been also proposed.[28]

In practice, nonsmooth nonlinearities such as deadzone, backlash, and hysteresis are commonly present in actuators and can significantly downgrade overall system performance. Adaptive inversion[29-35] or linearly parameterized models with disturbance-like modeling errors[36-43] have been employed in the literature to design controllers that alleviate the undesirable effects of input nonlinearities. Several applications have also been considered such as the synchronization control problem of chaotic systems with input nonlinearities.[44,45]

In this paper, we propose for the first time an adaptive control extension of the nonlinear PI methodology. The solution is provided for a class of uncertain strict-feedback nonlinear systems with nonsmooth input nonlinearities and unknown control directions. We adopt a linear time-varying approximate model derived in the work of Zheng et al[20] to describe a general class of nonsmooth input nonlinearities. The main contributions of this work are the following.

- We propose a new variant of the nonlinear PI method that *allows for parameter adaptation*. The new adaptive approach relaxes the standard assumption made in earlier nonlinear PI schemes[1,27] of known bounds for the associated system nonlinearities. This extension is not trivial and is made possible through the use of a new technical lemma (Lemma 1).
- We generalize the method to reference trajectory tracking, which was left as an open problem in remark 9 in the work of Ortega et al.[1] This is achieved by employing as an argument in the nonlinear PI terms *the square of the output of a deadzone with input the tracking error* and deadzone levels that are determined by the desired tracking accuracy.
- The new approach avoids the explosion of complexity problem since there is no need for explicit calculation of the partial derivatives of the virtual control laws. Suitable upper bounds depending on already known variables are obtained and estimated by the neural networks.
- A new integrator backstepping procedure is considered resulting in virtual control laws, which are calculated from suitable integral equations in the spirit of our other work.[27]

The rest of this paper is organized as follows. In Section 2, the problem under study is defined and some standard results on linear parametric approximators (LPAs) are revisited. In Section 3, the main technical lemma of this paper is stated and proved. Using this result, a detailed backstepping design is described in Section 4. Boundedness and approximate trajectory tracking for the proposed backstepping design are proved in Section 5. Finally, simulation results are given in Section 6.

2 | PRELIMINARIES AND PROBLEM FORMULATION

2.1 | Notations

For $\delta > 0$, we denote by $\lfloor x \rceil_\delta$ the output of a symmetric deadzone with input $x \in \mathbb{R}$ and deadzone level δ, ie,

$$\lfloor x \rceil_\delta := \begin{cases} x - \delta \operatorname{sgn}(x), & |x| \geq \delta \\ 0, & |x| < \delta, \end{cases} \tag{1}$$

with $\operatorname{sgn}(x)$ being the sign of x. From the definition of $\lfloor x \rceil_\delta$ and using the fact that $\operatorname{sgn}(\lfloor x \rceil_\delta) = \operatorname{sgn}(x)$ for all $|x| > \delta$, we obtain the following inequalities that will be employed repeatedly in the subsequent analysis:

$$|x| \leq |\lfloor x \rceil_\delta| + \delta \quad \forall x \in \mathbb{R} \tag{2}$$

$$x \lfloor x \rceil_\delta = |x| \, |\lfloor x \rceil_\delta| \geq \delta |\lfloor x \rceil_\delta| \quad \forall x \in \mathbb{R} \tag{3}$$

$$\lfloor x \rceil_\delta^2 \leq x \lfloor x \rceil_\delta \quad \forall x \in \mathbb{R}. \tag{4}$$

In addition, the smallest integer larger than or equal to x is denoted by $\lceil x \rceil$.

2.2 | Problem formulation

We consider the following class of strict-feedback nonlinear systems with nonsmooth actuator nonlinearities:

$$\begin{aligned} \dot{x}_i &= f_i(\bar{x}_i) + g_i(\bar{x}_i) x_{i+1}, \quad y = x_1 \\ \dot{x}_n &= f_n(\bar{x}_n) + g_n(\bar{x}_n) h(u,t) + d(t), \end{aligned} \tag{5}$$

where $\bar{x}_i := [x_1, \dots, x_i]^T \in \mathbb{R}^i$, $x := [x_1, \dots, x_n]^T \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the control input, and $y \in \mathbb{R}$ is the system output. Functions $f_i : \mathbb{R}^i \to \mathbb{R}$, $g_i : \mathbb{R}^i \to \mathbb{R}$ $(1 \le i \le n)$ are the system's unknown smooth nonlinearities, whereas $h(u,t)$ represents the nonsmooth actuator nonlinearity, and $d(t)$ is some bounded disturbance signal.

Assumption 1. Functions $g_i(\cdot)$ are continuous, and there exist unknown positive constants $g_{m,i} > 0$ such that $0 < g_{m,i} \le |g_i(\bar{x}_i)|$ for all $\bar{x}_i \in \mathbb{R}^i$ $(1 \le i \le n)$. Thus, functions $g_i(\cdot)$ have constant but unknown signs (*control directions*).

Assumption 2. (See the work of Zheng et al[20])
The function $h(u,t)$ is described as

$$h(u,t) = m(t)u + l(t), \tag{6}$$

where $m(t)$ is a time-varying bounded function that takes values in the closed interval $I = [m^-, m^+]$ with $0 \notin I$ and $l(t)$ is a time-varying bounded function, satisfying $\|l(t)\| \le l^*$ for an unknown constant l^*.

Remark 1. Assumption 1 is plausible since presupposing that all g_i are far from zero is a controllability condition for system (5). In addition, as proved in detail in the work of Zheng et al,[20] several models of typical nonsmooth actuator nonlinearities such as deadzone, backlash, and hysteresis satisfy Assumption 2.

Assumption 3. The perturbation term $d(t)$ represents an unknown bounded disturbance, ie, it holds true that $\|d(t)\| \le d^*$ for some unknown constant d^*.

Assumption 4. The reference trajectory $y_d(t)$ and its time derivative are bounded, ie, there exist unknown constants $y_{M0}, y_{M1} > 0$ such that $|y_d(t)| \le y_{M0}$, $|\dot{y}_d(t)| \le y_{M1}$ for all $t \ge 0$.

Our design objective is to impose approximate output tracking, ie, to select a control law such that $\lim_{t\to\infty} |y(t) - y_d(t)| \le \delta_1$ for an arbitrary continuously differentiable reference trajectory $y_d(t)$ and tracking accuracy $\delta_1 > 0$.

Remark 2. The main goal of this paper is the extension of the nonlinear PI approach within an adaptive control framework, which is to the best of the authors' knowledge, a new and unsolved challenging theoretical problem. We note that the tracking control problem under study can also be addressed with the use of Nussbaum functions as in other works.[20,22,46] Thus, the proposed approach can be seen as a valid new alternative method since, compared with the standard Nussbaum gain technique, the nonlinear PI approach has improved robustness properties with respect to certain types of unmodeled dynamics (for details, see example 8 in the work of Georgiou and Smith[24] and section 6.3 in the work of Ortega et al[1] and in our other works[25,26])

2.3 | Linear parametric approximators

It is well known[47-49] that any continuous nonlinear function $f_0 : \mathbb{R}^n \to \mathbb{R}$ can be approximated within some compact set $\Omega \subset \mathbb{R}^n$ with arbitrary precision by a suitably chosen LPA such as radial basis functions or high-order neural networks. Specifically, we can write

$$f_0(x) = \theta^{*T}\Phi(x) + \varepsilon(x) \quad \forall x \in \Omega \subset \mathbb{R}^n, \tag{7}$$

where $\Phi(x) \in \mathbb{R}^\ell$ is the basis vector, ℓ is the number of nodes in the hidden layer, $\varepsilon(x)$ is the optimal approximation error, and $\theta^* \in \mathbb{R}^\ell$ is the optimal weight vector defined as

$$\theta^* := \arg\min_{\theta\in\mathbb{R}^\ell} \sup_{x\in\Omega} \|f_0(x) - \theta^T\Phi(x)\|. \tag{8}$$

According to the universal approximation property of LPAs,[49] the optimal approximation error norm $\varepsilon_M := \sup_{x\in\Omega}\|\varepsilon(x)\|$ can become arbitrarily small if we select a sufficiently large number of nodes ℓ in the hidden layer and an appropriate basis vector to ensure a complete covering of the approximation region.

3 | MAIN LEMMA

Our results are based on the following boundedness Lemma, which is a generalization of Lemma 1 in our other work.[27] This lemma is central for the analysis of the adaptive backstepping nonlinear PI schemes proposed in Section 4.

Lemma 1. *Consider the continuous differentiable nonnegative functions $V_i, L_i : [0, t_f) \to \mathbb{R}_+$ and the continuous functions $Q_i : [0, t_f) \to \mathbb{R}$ $(1 \le i \le n)$ and $P_i, \overline{P}_i : [0, t_f) \to \mathbb{R}$ defined as*

$$P_i(t) := V_i(t) + L_i(t) + \lambda_i \int_0^t V_i(s)ds - \varepsilon_{i+1}\lambda_{i+1} \int_0^t V_{i+1}(s)ds \tag{9}$$

$$\overline{P}_i(t) := P_i(t) + Q_i(t) \qquad (1 \le i \le n), \tag{10}$$

with $\varepsilon_i, \lambda_i > 0$. We also adopt the notation $V_{n+1} \equiv 0$ to ensure uniformity. Assume n continuous functions $g_i : [0, \infty) \to \mathbb{R}$ for which there exist positive constants $g_{m,i} > 0$ such that $0 < g_{m,i} \le |g_i(t)|$ for all $t \in [0, \infty)$. If there exist nonnegative functions $\Theta_i : [0, t_f) \to \mathbb{R}_+$ such that

$$\frac{d\overline{P}_i}{dt} \le [\eta_i^* + g_i(t)\kappa_i(P_i)]\Theta_i(t) \tag{11}$$

$$\int_{t_1}^{t_2} \Theta_i(t)dt \ge c_i|Q(t_2) - Q(t_1)| \qquad \forall 0 \le t_1 \le t_2 < t_f, \tag{12}$$

with constants $c_i > 0$, $\eta_i^* \ge 0$, $\kappa_i(P_i) := \varphi_i(P_i^2)\cos(P_i)$ and $\varphi_i(\cdot)$ a class $\mathcal{K}_\infty$ function, then P_i is upper bounded on $[0, t_f)$. Moreover, if $\varepsilon_i < 1$ for all $i = 1, \dots, n$, then all $V_i, L_i, \int_0^t V_i(s)ds$ are bounded on $[0, t_f)$.

Proof. The *upper boundedness* of P_i can be proved by using a contradiction argument. If we assume that P_i has no upper bound, then due to continuity of P_i, there exist 2 sequences of time instants $\{t_{i,1k}\}$ and $\{t_{i,2k}\}$ defined as

$$t_{i,2k} = \inf\left\{ t \in [0, t_f) | P_i(t) = 2k\pi + \frac{\pi}{2}\text{sgn}(g_i) + \frac{3\pi}{4} \right\} \tag{13}$$

$$t_{i,1k} = \sup\left\{ t \in [0, t_{2k}) | P_i(t) = 2k\pi + \frac{\pi}{2}\text{sgn}(g_i) + \frac{\pi}{4} \right\} \tag{14}$$

such that

$$g_i(t)\kappa_i(P_i(t)) \le -\frac{g_{m,i}}{\sqrt{2}}\varphi_i((2k\pi - \pi/4)^2),\ \forall t \in [t_{i,1k}, t_{i,2k}]. \tag{15}$$

Integrating (11) over $[t_{i,1k}, t_{i,2k}]$, we obtain

$$\overline{P}_i(t_{i,2k}) - \overline{P}_i(t_{i,1k}) \le \int_{t_{i,1k}}^{t_{i,2k}} [\eta_i^* + g_i(t)\kappa_i(P_i)]\Theta_i(t)dt. \tag{16}$$

Using now property (15), the aforementioned inequality yields

$$\overline{P}_i(t_{i,2k}) - \overline{P}_i(t_{i,1k}) \le -\left[\frac{g_{m,i}}{\sqrt{2}}\varphi_i((2k\pi - \pi/4)^2) - \eta_i^*\right] \int_{t_{i,1k}}^{t_{i,2k}} \Theta_i(t)dt. \tag{17}$$

For sufficiently large $k \ge \left\lceil (1/2\pi)\left(\left(\varphi_i^{-1}\left(\sqrt{2}\eta_i^*/g_{m,i}\right)\right)^{1/2} + \pi/4\right)\right\rceil$, we have from the aforementioned inequality and (12) that

$$\overline{P}_i(t_{i,2k}) - \overline{P}_i(t_{i,1k}) \le -c_i\left[\frac{g_{m,i}}{\sqrt{2}}\varphi_i((2k\pi - \pi/4)^2) - \eta_i^*\right] |Q_i(t_{i,2k}) - Q_i(t_{i,1k})|. \tag{18}$$

Moreover, from the definition of $\overline{P}_i$ in (10) and (13), (14), we have that

$$\begin{aligned}\overline{P}_i(t_{i,2k}) - \overline{P}_i(t_{i,1k}) &= P_i(t_{i,2k}) - P_i(t_{i,1k}) + Q_i(t_{i,2k}) - Q_i(t_{i,1k}) \\ &= Q_i(t_{i,2k}) - Q_i(t_{i,1k}) + \pi/2 \\ &\ge -|Q_i(t_{i,2k}) - Q_i(t_{i,1k})| + \pi/2.\end{aligned} \tag{19}$$

Combining (18), (19), we obtain

$$\frac{\pi}{2} + \left[c_i\frac{g_{m,i}}{\sqrt{2}}\varphi_i((2k\pi - \pi/4)^2) - c_i\eta_i^* - 1\right] |Q_i(t_{i,2k}) - Q_i(t_{i,1k})| \le 0, \tag{20}$$

which cannot hold for arbitrarily large k. Specifically, (20) does not hold for

$$k \ge \overline{k}_i := \left\lceil \frac{1}{2\pi}\left[\left(\varphi_i^{-1}\left(\frac{\sqrt{2}}{c_i g_{m,i}}(c_i\eta_i^* + 1)\right)\right)^{1/2} + \frac{\pi}{4}\right]\right\rceil. \tag{21}$$

Thus, considering the integer $k_{i,0} := \inf\{k \in \mathbb{N} | P_i(0) \le 2k\pi + (\pi/2)\mathrm{sgn}(g_i) + \pi/4\}$ described as

$$k_{i,0} := \left\lceil \frac{1}{2\pi}\left[P_i(0) - \frac{\pi}{4} - \frac{\pi}{2}\mathrm{sgn}(g_i)\right]\right\rceil, \tag{22}$$

we have that P_i is upper bounded in $[0, t_f)$ from

$$P_i(t) \le 2\max\{k_{i,0}, \bar{k}_i\}\pi + \frac{\pi}{2}\mathrm{sgn}(g_i) + \frac{3\pi}{4}. \tag{23}$$

If $\varepsilon_i < 1$ for all $i = 1, \dots, n$, then using the upper boundedness of all P_i, we can sum all (9) to obtain

$$\sum_{i=1}^{n} P_i(t) = \sum_{i=1}^{n}(V_i + L_i) + \sum_{i=1}^{n}(1-\varepsilon_i)\lambda_i \int_0^t V_i(s)ds \ge 0. \tag{24}$$

The aforementioned identity and the boundedness of $\sum_{i=1}^{n} P_i(t)$ yield the boundedness of all $L_i, V_i, \int_0^t V_i(s)ds$ on $[0, t_f)$. □

4 | BACKSTEPPING CONTROL DESIGN

In this section, we provide a backstepping design procedure that allows for an application of the boundedness Lemma 1. Approximate output tracking can then be proved using the conclusions of Lemma 1.

4.1 | Step 1

If we define the first error variable $z_1 := x_1 - y_d$, then

$$\dot{z}_1 = f_1(x_1) + g_1(x_1)x_2 - \dot{y}_d. \tag{25}$$

Let α_1 be the virtual control to be selected and $z_2 := x_2 - \alpha_1$ be the second error variable. For the nonnegative continuous differentiable function $V_1 := (1/2)\lfloor z_1 \rceil_{\delta_1}^2$ with $\delta_1 > 0$, we have

$$\dot{V}_1 = \lfloor z_1 \rceil_{\delta_1}\dot{z}_1 = \lfloor z_1 \rceil_{\delta_1}\left[f_1(x_1) + g_1(x_1)z_2 - \dot{y}_d + g_1(x_1)\alpha_1\right]. \tag{26}$$

Using (2), (3) and completing the square, we obtain the following bound:

$$\begin{aligned} g_1(x_1)z_2\lfloor z_1 \rceil_{\delta_1} &\le |g_1(x_1)||z_2| \left|\lfloor z_1 \rceil_{\delta_1}\right| \\ &\le |g_1(x_1)|(\left|\lfloor z_2 \rceil_{\delta_2}\right| + \delta_2)\left|\lfloor z_1 \rceil_{\delta_1}\right| \\ &\le \delta_2\left|g_1(x_1)\lfloor z_1 \rceil_{\delta_1}\right| + \frac{1}{2\varepsilon_2\lambda_2}g_1^2(x_1)\lfloor z_1 \rceil_{\delta_1}^2 + \frac{\varepsilon_2\lambda_2}{2}\lfloor z_2 \rceil_{\delta_2}^2 \\ &\le \frac{\delta_2}{\delta_1}|g_1(x_1)|z_1\lfloor z_1 \rceil_{\delta_1} + \frac{1}{2\varepsilon_2\lambda_2}g_1^2(x_1)\lfloor z_1 \rceil_{\delta_1}^2 + \frac{\varepsilon_2\lambda_2}{2}\lfloor z_2 \rceil_{\delta_2}^2 \end{aligned} \tag{27}$$

for some $\delta_2, \lambda_2 > 0$ and $0 < \varepsilon_2 < 1$. From (26) and the aforementioned inequality, we result in

$$\dot{V}_1 \le \lfloor z_1 \rceil_{\delta_1}[F_1(x_1, z_1) - \dot{y}_d + g_1(x_1)\alpha_1] + \frac{\varepsilon_2\lambda_2}{2}\lfloor z_2 \rceil_{\delta_2}^2 \tag{28}$$

with

$$F_1(x_1, z_1) := f_1(x_1) + \frac{\delta_2}{\delta_1}|g_1(x_1)|z_1 + \frac{1}{2\varepsilon_2\lambda_2}g_1^2(x_1)\lfloor z_1 \rceil_{\delta_1}. \tag{29}$$

Consider now an approximation of the nonlinearity F_1 within some compact set $\Omega_1 \subset \mathbb{R}^2$ by an LPA such that

$$F_1(x_1, z_1) = \theta_1^{*T}\Phi_1(x_1, z_1) + \varepsilon_{a1}(x_1, z_1),\ \forall (x_1, z_1) \in \Omega_1, \tag{30}$$

with $\theta_1^* \in \mathbb{R}^{\ell_1}$ being the optimal approximation weight, $\Phi_1(x_1, z_1) \in \mathbb{R}^{\ell_1}$ being the regressor vector, and $\varepsilon_{a1}(x_1, z_1)$ being the approximation error. The error is bounded within Ω_1, ie, there exists some $\varepsilon_{M1} > 0$ such that $|\varepsilon_{a1}(x_1, z_1)| \le \varepsilon_{M1}$ for all $(x_1, z_1) \in \Omega_1$. In our control law, we consider now estimations of the optimal weight norm instead of the whole weight vector. This is an idea originally introduced in the work of Chen et al[50] and used also in the work of Wang et al[51] to reduce the overall computational cost. If we introduce an estimation β_1 of the optimal weight norm $\|\theta_1^*\|$ and the functions $L_1 := (1/2\gamma_1)\beta_1^2$,

$$P_1 := V_1 + L_1 + \lambda_1 \int_0^t V_1(s)ds - \frac{\varepsilon_2\lambda_2}{2}\int_0^t \lfloor z_2(s)\rceil_{\delta_2}^2 ds \tag{31}$$

with $\gamma_1, \lambda_1 > 0$, then

$$\dot{P}_1 \le \lfloor z_1\rceil_{\delta_1}\left[\theta_1^{*T}\Phi_1(x_1,z_1) + \varepsilon_{a1}(x_1,z_1) + \frac{\lambda_1}{2}\lfloor z_1\rceil_{\delta_1} - \dot{y}_d + g_1(x_1)\alpha_1\right] + \frac{1}{\gamma_1}\beta_1\dot{\beta}_1. \tag{32}$$

Define also the functions $Q_1 := (1/2\gamma_1)(\tilde{\beta}_1^2 - \beta_1^2)$ with $\tilde{\beta_1} := \beta_1 - \|\theta_1^*\|$ as the weight norm estimation error and

$$\overline{P}_1 := P_1 + Q_1. \tag{33}$$

If we select the estimation update law,

$$\dot{\beta}_1 = \gamma_1\|\Phi_1(x_1,z_1)\|\,|\lfloor z_1\rceil_{\delta_1}|, \quad \beta_1(0) = 0, \tag{34}$$

then from (32)-(34) and Assumption 4, we have

$$\begin{aligned}\frac{d\overline{P}_1}{dt} &\le \lfloor z_1\rceil_{\delta_1}\left[\theta_1^{*T}\Phi_1(x_1,z_1) + \varepsilon_{a1}(x_1,z_1) + \frac{\lambda_1}{2}\lfloor z_1\rceil_{\delta_1} - \dot{y}_d + g_1(x_1)\alpha_1\right] + (\beta_1 - \|\theta_1^*\|)\|\Phi_1\|\,|\lfloor z_1\rceil_{\delta_1}| \\ &\le |\lfloor z_1\rceil_{\delta_1}|\left[\beta_1\|\Phi_1(x_1,z_1)\| + \varepsilon_{M1} + y_{M1}\right] + \frac{1}{2}\lambda_1\lfloor z_1\rceil_{\delta_1}^2 + g_1(x_1)\lfloor z_1\rceil_{\delta_1}\alpha_1.\end{aligned} \tag{35}$$

Using (3), the aforementioned inequality yields

$$\frac{d\overline{P}_1}{dt} \le \frac{1}{\delta_1}\lfloor z_1\rceil_{\delta_1}\left[\beta_1\|\Phi_1(x_1,z_1)\|z_1 + (\varepsilon_{M1} + y_{M1})z_1 + \frac{1}{2}\delta_1\lambda_1\lfloor z_1\rceil_{\delta_1} + \delta_1 g_1(x_1)\alpha_1\right]. \tag{36}$$

If we now select the virtual control law,

$$\alpha_1 = \kappa_1(P_1)\left(1 + \|\Phi_1(x_1,z_1)\|^2 + \beta_1^2\right)z_1 \tag{37}$$

along with (4) and consider the following inequality obtained by completing the square:

$$\beta_1\|\Phi_1(x_1,z_1)\| \le \frac{1}{2}(\|\Phi_1(x_1,z_1)\|^2 + \beta_1^2), \tag{38}$$

then (36) yields

$$\frac{d\overline{P}_1}{dt} \le [\eta_1^* + g_1(x_1(t))\kappa_1(P_1)]\Theta_1(t) \tag{39}$$

with

$$\Theta_1(t) := \left(1 + \|\Phi_1(x_1,z_1)\|^2 + \beta_1^2\right)z_1\lfloor z_1\rceil_{\delta_1} \ge 0 \tag{40}$$

$$\eta_1^* := \max\left\{\frac{1}{2\delta_1}, \frac{\lambda_1}{2} + \frac{\varepsilon_{M1} + y_{M1}}{\delta_1}\right\}. \tag{41}$$

From the definition of Q_1 and (34), (3), (40), we obtain

$$\begin{aligned}|Q_1(t_1) - Q_1(t_2)| &= \frac{1}{\gamma_1}\|\theta_1^*\|\;|\beta_1(t_2) - \beta_1(t_1)| \\ &= \frac{1}{\gamma_1}\|\theta_1^*\|\left|\int_{t_1}^{t_2}\dot{\beta}_1(s)ds\right| = \int_{t_1}^{t_2}\|\theta_1^*\|\;\|\Phi_1(x_1,z_1)\|\,|\lfloor z_1\rceil_{\delta_1}|ds \\ &\le \frac{1}{2\delta_1}\int_{t_1}^{t_2}\left(\|\theta_1^*\|^2 + \|\Phi_1(x_1,z_1)\|^2\right)z_1\lfloor z_1\rceil_{\delta_1}ds \\ &\le \frac{\max\left\{1, \|\theta_1^*\|^2\right\}}{2\delta_1}\int_{t_1}^{t_2}\Theta_1(s)ds \quad, \forall t_2 \ge t_1.\end{aligned} \tag{42}$$

Thus, from (39), (42), conditions (11), (12) of Lemma 1 are true, and therefore, the function P_1 is upper bounded.

Similarly to our other work,[27] we note that (37) defining the virtual control α_1 is actually an integral equation of the form

$$\alpha_1(t) = \kappa_1 \left[V_1 + L_1 - \frac{\varepsilon_2\lambda_2}{2} \int_0^t \lfloor x_2(s) - \alpha_1(s) \rceil_{\delta_2}^2 ds + \lambda_1 \int_0^t V_1(s) ds \right] \left(1 + \|\Phi_1(x_1, z_1)\|^2 + \beta_1^2\right) z_1. \tag{43}$$

Alternatively, the virtual control law α_1 can be seen as the output of a first-order dynamical system with state vector

$$\zeta_1 := \frac{\lambda_1}{2} \int_0^t \lfloor z_1(s) \rceil_{\delta_1}^2 ds - \frac{\varepsilon_2\lambda_2}{2} \int_0^t \lfloor z_2(s) \rceil_{\delta_2}^2 ds \tag{44}$$

and input vector $[z_1, \beta_1, x_1, x_2]$. This follows from the state and output equations,

$$\dot{\zeta}_1 = \frac{\lambda_1}{2} \lfloor z_1 \rceil_{\delta_1}^2 - \frac{\varepsilon_2\lambda_2}{2} \left\lfloor x_2 - \kappa_1 \left(\frac{1}{2} \lfloor z_1 \rceil_{\delta_1}^2 + \frac{1}{2\gamma_1}\beta_1^2 + \zeta_1 \right) \left(1 + \|\Phi_1(x_1, z_1)\|^2 + \beta_1^2\right) z_1 \right\rceil_{\delta_2}^2 \tag{45}$$

$$\alpha_1 = \kappa_1 \left(\frac{1}{2} \lfloor z_1 \rceil_{\delta_1}^2 + \frac{1}{2\gamma_1}\beta_1^2 + \zeta_1 \right) \left(1 + \|\Phi_1(x_1, z_1)\|^2 + \beta_1^2\right) z_1. \tag{46}$$

4.2 | Step i ($2 \le i \le n-1$):

For the ith step, we define the error variable $z_i := x_i - \alpha_{i-1}$ with dynamics,

$$\dot{z}_i = f_i(\overline{x}_i) + g_i(\overline{x}_i) z_{i+1} + g_i(\overline{x}_i)\alpha_i - \dot{\alpha}_{i-1}, \tag{47}$$

where α_i is the ith virtual control. The following lemma holds true.

Lemma 2. *Consider the nonlinear system (5) the error variables $z_1 = x_1 - y_d$, $z_i = x_i - \alpha_{i-1}$, $z_{n+1} \equiv 0$, and the integral terms defined by:*

$$I_i = \int_0^t \lfloor z_i \rceil_{\delta_i}^2 (s) ds. \tag{48}$$

If all virtual control laws are selected to have the form

$$\alpha_i = A_i(\overline{x}_i, \overline{z}_i, \overline{I}_{i+1}, \overline{\beta}_i), \quad (1 \le i \le n-1), \tag{49}$$

where β_i are adaptation parameters defined in each step of the backstepping procedure with the property,

$$|\dot{\beta}_i| \le B_i(\overline{x}_i, \overline{z}_i, \overline{I}_i, \overline{\beta}_{i-1}) \tag{50}$$

and $\xi_i := [\overline{x}_i, \overline{z}_i, \overline{I}_i, \overline{\beta}_{i-1}]$, $B_i \circ \xi_i \in C^1([t_0, t_f), \mathbb{R}_+)$, then there exist continuous differentiable functions $H_i \circ \xi_{i+1} \in C^1([t_0, t_f), \mathbb{R}_+)$ such that

$$|\dot{\alpha}_i| \le H_i(\overline{x}_{i+1}, \overline{z}_{i+1}, \overline{I}_{i+1}, \overline{\beta}_i), \quad (1 \le i \le n-1). \tag{51}$$

Proof. We will employ an induction argument to prove (51). For $i = 1$, we have from (49), (5) that

$$\dot{\alpha}_1 = \frac{\partial A_1}{\partial x_1}(f_1 + g_1 x_2) + \frac{\partial A_1}{\partial z_1}(f_1 + g_1 x_2 - \dot{y}_d) + \sum_{j=1}^{2} \frac{\partial A_1}{\partial I_j} \lfloor z_j \rceil_{\delta_j}^2 + \frac{\partial A_1}{\partial \beta_1}\dot{\beta}_1. \tag{52}$$

Using (50) and completing the square, the aforementioned inequality yields

$$\begin{aligned} |\dot{\alpha}_1| &\le \left| \frac{\partial A_1}{\partial x_1}(f_1 + g_1 x_2) \right| + \left| \frac{\partial A_1}{\partial z_1} \right| (|f_1 + g_1 x_2| + y_{M1}) + \sum_{j=1}^{2} \left| \frac{\partial A_1}{\partial I_j} \right| \lfloor z_j \rceil_{\delta_j}^2 + \left| \frac{\partial A_1}{\partial \beta_1} \right| B_1 \\ &\le \frac{1}{2}\left(\frac{\partial A_1}{\partial x_1} \right)^2 + (f_1 + g_1 x_2)^2 + \left(\frac{\partial A_1}{\partial z_1} \right)^2 + \frac{y_{M1}^2}{2} + \frac{1}{2}\left(\frac{\partial A_1}{\partial \beta_1} \right)^2 \\ &\quad + \frac{1}{2}\sum_{j=1}^{2} \left[\left(\frac{\partial A_1}{\partial I_j} \right)^2 + \lfloor z_j \rceil_{\delta_j}^4 \right] + \frac{1}{2}B_1^2 := H_1(\overline{x}_2, \overline{z}_2, \overline{I}_2, \beta_1), \end{aligned} \tag{53}$$

ie, (51) is valid for $i = 1$. Suppose that (51) holds true for all $j = 1, 2, \dots, i-1$. We will prove that (51) is also valid for $j = i$. Taking the time derivative of (49), we have that

$$\dot{\alpha}_1 = \sum_{j=1}^{i} \frac{\partial A_i}{\partial x_j}\dot{x}_j + \sum_{j=1}^{i} \frac{\partial A_i}{\partial z_j}\dot{z}_j + \sum_{j=1}^{i+1} \frac{\partial A_i}{\partial I_j}\dot{I}_j + \sum_{j=1}^{i} \frac{\partial A_i}{\partial \beta_j}\dot{\beta}_j. \tag{54}$$

From (5), it holds true that $\dot{x}_i = f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1} := h_{x,i}(\bar{x}_{i+1})$ for all $i = 1, \dots, n-1$. Since we assumed that (51) is true for all $j = 1, \dots, i-1$, we can write

$$\begin{aligned} |\dot{z}_j| \le |h_{x,j}(\bar{x}_{j+1})| + |\dot{\alpha}_{j-1}| &\le h^2_{x,j}(\bar{x}_{j+1}) + H_{j-1}(\bar{x}_j, \bar{z}_j, \bar{I}_j, \bar{\beta}_{j-1}) + \frac{1}{4} \\ &:= h_{z,j}(\bar{x}_{j+1}, \bar{z}_j, \bar{I}_j, \bar{\beta}_{j-1}) \quad \forall j = 1, \dots, i. \end{aligned} \tag{55}$$

Using (55) and (50) in (54), we obtain

$$\begin{aligned} |\dot{\alpha}_i| &\le \sum_{j=1}^{i} \left| \frac{\partial A_i}{\partial x_j} h_{x,j} \right| + \sum_{j=1}^{i} \left| \frac{\partial A_i}{\partial z_j} \right| h_{z,j} + \sum_{j=1}^{i+1} \left| \frac{\partial A_i}{\partial I_j} \right| \lfloor z_j \rceil^2_{\delta_j} + \sum_{j=1}^{i} \left| \frac{\partial A_i}{\partial \beta_j} \right| B_j \\ &\le \frac{1}{2}\sum_{j=1}^{i} \left[\left(\frac{\partial A_i}{\partial x_j} \right)^2 + \left(\frac{\partial A_i}{\partial z_j} \right)^2 + \left(\frac{\partial A_i}{\partial \beta_j} \right)^2 + h^2_{x,j} + h^2_{z,j} + B^2_j \right] \\ &\quad + \frac{1}{2}\sum_{j=1}^{i+1} \left[\left(\frac{\partial A_i}{\partial I_j} \right)^2 + 1 \right] \lfloor z_j \rceil^2_{\delta_j} := H_i(\bar{x}_{i+1}, \bar{z}_{i+1}, \bar{I}_{i+1}, \bar{\beta}_i), \end{aligned} \tag{56}$$

which completes the proof of the lemma. □

Obviously, α_1 defined in (37) is of the form (49). Moreover, the dynamics of β_1 given by (34) ensure that

$$|\dot{\beta}_1| \le \frac{\gamma_1}{2} \left(\|\Phi_1(x_1, z_1)\|^2 + \lfloor z_1 \rceil^2_{\delta_1} \right) := B_1(x_1, z_1, I_1), \tag{57}$$

which is in the form of (50). Assume that the virtual control law α_j and the dynamics of β_j are in the form of (49), (50), respectively, for all $j = 1, 2, \dots, i-1$. In the following, we will select a virtual control α_i and an estimate β_i for which (49), (50) are also true.

Consider now the function $V_i = \frac{1}{2}\lfloor z_i \rceil^2_{\delta_i}$ with $\delta_i > 0$. The time derivative of V_i is

$$\dot{V}_i = \lfloor z_i \rceil_{\delta_i} [f_i(\bar{x}_i) + g_i(\bar{x}_i)z_{i+1} + g_i(\bar{x}_i)\alpha_i - \dot{\alpha}_{i-1}]. \tag{58}$$

Using Lemma 2 and (3), we have that

$$\dot{V}_i \le \lfloor z_i \rceil_{\delta_i} \left[f_i(\bar{x}_i) + g_i(\bar{x}_i)z_{i+1} + g_i(\bar{x}_i)\alpha_i + (1/\delta_i) z_i H_{i-1}\left(\bar{x}_i, \bar{z}_i, \bar{I}_i, \bar{\beta}_{i-1}\right) \right]. \tag{59}$$

Similar to the derivation of (27) in step $i = 1$, we can prove that $g_i(\bar{x}_i)z_{i+1}\lfloor z_i \rceil_{\delta_i}$ is bounded by

$$g_i(\bar{x}_i)z_{i+1}\lfloor z_i \rceil_{\delta_i} \le \frac{\delta_{i+1}}{\delta_i}|g_i(\bar{x}_i)| z_i \lfloor z_i \rceil_{\delta_i} + \frac{1}{2\varepsilon_{i+1}\lambda_{i+1}} g_i^2(\bar{x}_i)\lfloor z_i \rceil^2_{\delta_i} + \frac{\varepsilon_{i+1}\lambda_{i+1}}{2}\lfloor z_{i+1} \rceil^2_{\delta_{i+1}} \tag{60}$$

for $\delta_{i+1}, \lambda_{i+1} > 0$ and $0 < \varepsilon_{i+1} < 1$. If we define $\xi_i := [\bar{x}_i, \bar{z}_i, \bar{I}_i, \bar{\beta}_{i-1}] \in \mathbb{R}^{4i-1}$ and

$$F_i(\xi_i) := f_i(\bar{x}_i) + \frac{\delta_{i+1}}{\delta_i}|g_i(\bar{x}_i)| z_i + \frac{1}{2\varepsilon_{i+1}\lambda_{i+1}} g_i^2(\bar{x}_i)\lfloor z_i \rceil_{\delta_i} + \frac{1}{\delta_i} z_i H_{i-1}(\xi_i), \tag{61}$$

then (59) yields the following through (60), (61):

$$\dot{V}_i \le \lfloor z_i \rceil_{\delta_i}[F_i(\xi_i) + g_i(\bar{x}_i)\alpha_i] + \frac{\varepsilon_{i+1}\lambda_{i+1}}{2}\lfloor z_{i+1} \rceil^2_{\delta_{i+1}}. \tag{62}$$

Consider now an approximation of the nonlinearity $F_i(\xi_i)$ within some compact set $\Omega_i \subset \mathbb{R}^{4i-1}$ by an LPA such that

$$F_i(\xi_i) := \theta_i^{*T}\Phi_i(\xi_i) + \varepsilon_{ai}(\xi_i), \quad \forall \xi_i \in \Omega_i, \tag{63}$$

with $\theta_i^* \in \mathbb{R}^{\ell_i}$ being the optimal approximation weight, $\Phi_i(\xi_i) \in \mathbb{R}^{\ell_i}$ being the regressor vector, and $\varepsilon_{ai}(\xi_i)$ being the approximation error. The error is bounded within Ω_i, ie, there exists some $\varepsilon_{Mi} > 0$ such that $|\varepsilon_{ai}(\xi_i)| \le \varepsilon_{Mi}$ for all $\xi_i \in \Omega_i$.

If we introduce an estimation β_i of the optimal weight norm $\|\theta_i^*\|$ and functions $L_i := (1/2\gamma_i)\beta_i^2$,

$$P_i := V_i + L_i + \lambda_i \int_0^t V_i(s)ds - \frac{\varepsilon_{i+1}\lambda_{i+1}}{2} \int_0^t \lfloor z_{i+1}(s) \rceil_{\delta_{i+1}}^2 ds \tag{64}$$

with $\gamma_i > 0$, then from (62), (63), we have

$$\dot{P}_i \leq \lfloor z_i \rceil_{\delta_i} \left[\theta_i^{*T} \Phi_i(\xi_i) + \varepsilon_{ai}(\xi_i) + (\lambda_i/2) \lfloor z_i \rceil_{\delta_i} + g_i(\bar{x}_i)\alpha_i \right] + \frac{1}{\gamma_i}\beta_i \dot{\beta}_i. \tag{65}$$

Let us define the functions $Q_i := (1/2\gamma_i)(\tilde{\beta}_i^2 - \beta_i^2)$, where $\tilde{\beta}_i = \beta_i - \|\theta_i^*\|$ and

$$\overline{P}_i := P_i + Q_i. \tag{66}$$

If we select the estimation update law,

$$\dot{\beta}_i = \gamma_i \|\Phi_i(\xi_i)\| |\lfloor z_i \rceil_{\delta_i}|, \quad \beta_i(0) = 0, \tag{67}$$

then

$$|\dot{\beta}_i| \leq B_i(\xi_i) := \frac{\gamma_i}{2} \left[\|\Phi_i(\xi_i)\|^2 + \lfloor z_i \rceil_{\delta_i}^2 \right], \tag{68}$$

which is of the form (50) of Lemma 2. In addition, from (65)-(67), (3), we have

$$\frac{d\overline{P}_i}{dt} \leq \lfloor z_i \rceil_{\delta_i} \left[\frac{\lambda_i}{2} \lfloor z_i \rceil_{\delta_i} + \frac{\varepsilon_{Mi}}{\delta_i} z_i + g_i(\bar{x}_i)\alpha_i \right] + \beta_i \|\Phi_i(\xi_i)\| |\lfloor z_i \rceil_{\delta_i}|. \tag{69}$$

Completing the square and using (3), we result in

$$\beta_i \|\Phi_i(\xi_i)\| |\lfloor z_i \rceil_{\delta_i}| \leq \frac{1}{2\delta_i} \left(\beta_i^2 + \|\Phi_i(\xi_i)\|^2 \right) \lfloor z_i \rceil_{\delta_i} z_i. \tag{70}$$

If we choose the virtual control law,

$$\alpha_i = \kappa_i(P_i) \left(1 + |\beta_i|^2 + \|\Phi_i(\xi_i)\|^2 \right) z_i := A_i \left(\bar{x}_i, \bar{z}_i, \bar{I}_{i+1}, \bar{\beta}_i \right), \tag{71}$$

which is of the form (49), then from (69), (70) and (4), we obtain

$$\frac{d\overline{P}_i}{dt} \leq \left[\eta_i^* + g_i(\bar{x}_i(t))\kappa_i(P_i) \right] \Theta_i(t) \tag{72}$$

with

$$\Theta_i(t) := \left(1 + \beta_i^2 + \|\Phi_i(\xi_i)\|^2 \right) z_i \lfloor z_i \rceil_{\delta_i} \tag{73}$$

$$\eta_i^* := \max \left\{ \frac{1}{2\delta_i}, \frac{\lambda_i}{2} + \frac{\varepsilon_{Mi}}{\delta_i} \right\}. \tag{74}$$

From the definition of Q_i and (67), (3), (73), it also holds true that

$$\begin{aligned} |Q_i(t_2) - Q_i(t_1)| &= \frac{1}{\gamma_i} \|\theta_i^*\| \, |\beta_i(t_2) - \beta_i(t_1)| \\ &= \frac{1}{\gamma_i} \|\theta_i^*\| \left| \int_{t_1}^{t_2} \dot{\beta}_i(s)ds \right| = \int_{t_1}^{t_2} \|\theta_i^*\| \, \|\Phi_i(\xi_i)\| |\lfloor z_i \rceil_{\delta_i}| ds \\ &\leq \frac{1}{2\delta_i} \int_{t_1}^{t_2} \left(\|\theta_i^*\|^2 + \|\Phi_i(\xi_i)\|^2 \right) z_i \lfloor z_i \rceil_{\delta_i} ds \\ &\leq \frac{\max\left\{1, \|\theta_i^*\|^2\right\}}{2\delta_i} \int_{t_1}^{t_2} \Theta_i(s)ds \quad \forall t_2 \geq t_1. \end{aligned} \tag{75}$$

Thus, conditions of Lemma 1 hold true, and therefore, P_i is upper bounded.

We also note that (71) defining the virtual control α_i is actually an integral equation of the form as follows:

$$\alpha_i(t) = \kappa_i \left[V_i + L_i - \frac{\varepsilon_{i+1}\lambda_{i+1}}{2} \int_0^t \lfloor x_{i+1}(s) - \alpha_i(s) \rceil_{\delta_{i+1}}^2 ds + \lambda_i \int_0^t V_i(s)ds \right] \left(1 + \beta_i^2 + \|\Phi_i(\xi_i)\|^2 \right) z_i. \tag{76}$$

Similar to the case $i = 1$, the virtual control law α_i can be interpreted as the output of a first-order dynamical system with state as follows:

$$\zeta_i := \frac{\lambda_i}{2}\int_0^t \lfloor z_i(s)\rceil_{\delta_i}^2 ds - \frac{\varepsilon_{i+1}\lambda_{i+1}}{2}\int_0^t \lfloor z_{i+1}(s)\rceil_{\delta_{i+1}}^2 ds \tag{77}$$

and input vector $[z_i, \beta_i, \xi_i^T, x_{i+1}]^T$. This follows from the state-space and output equations given by

$$\dot{\zeta}_i = \frac{\lambda_i}{2}\lfloor z_i\rceil_{\delta_i}^2 - \frac{\varepsilon_{i+1}\lambda_{i+1}}{2}\left\lfloor x_{i+1} - \kappa_i\left(\frac{1}{2}\lfloor z_i\rceil_{\delta_i}^2 + \frac{1}{2\gamma_i}\beta_i^2 + \zeta_i\right)\left(1 + \beta_i^2 + \|\Phi_i(\xi_i)\|^2\right) z_i\right\rceil_{\delta_{i+1}}^2 \tag{78}$$

$$\alpha_i = \kappa_i\left(\frac{1}{2}\lfloor z_i\rceil_{\delta_i}^2 + \frac{1}{2\gamma_i}\beta_i^2 + \zeta_i\right)\left(1 + \beta_i^2 + \|\Phi_i(\xi_i)\|^2\right) z_i. \tag{79}$$

The complexity in the calculation of the ith virtual control is, in this sense, comparable with standard Nussbaum schemes.[46] In the work of Zhang et al,[46] for example, the calculation of the ith virtual control law involves the ith Nussbaum parameter, which is generated by a first-order dynamical system.

4.3 | Step n

For the nth step, we define the error variable $z_n := x_n - \alpha_{n-1}$ with dynamics

$$\dot{z}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n)h(u,t) + d(t) - \dot{\alpha}_{n-1}. \tag{80}$$

Using Assumption 2, Equation (80) can be rewritten in the following form:

$$\dot{z}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n)m(t)\alpha_n + g_n(\bar{x}_n)l(t) + d(t) - \dot{\alpha}_{n-1}, \tag{81}$$

with $u = \alpha_n$ being the control law to be selected.

Consider now the function $V_n = \frac{1}{2}\lfloor z_n\rceil_{\delta_n}^2$ with $\delta_n > 0$. The time derivative of V_n is

$$\dot{V}_n = \lfloor z_n\rceil_{\delta_n}\left[f_n(\bar{x}_n) + g_n(\bar{x}_n)m(t)\alpha_n + g_n(\bar{x}_n)l(t) + d(t) - \dot{\alpha}_{n-1}\right]. \tag{82}$$

Using Lemma 2, Assumption 2, and (3), we have that

$$\dot{V}_n \le \lfloor z_n\rceil_{\delta_n}\left[f_n(\bar{x}_n) + g_n(\bar{x}_n)m(t)\alpha_n + d(t) + \frac{z_n}{\delta_n}\left(|g_n(\bar{x}_n)|l^* + H_{n-1}\left(\bar{x}_n, \bar{z}_n, \bar{I}_n, \bar{\beta}_{n-1}\right)\right)\right]. \tag{83}$$

Let $\xi_n := [\bar{x}_n, \bar{z}_n, \bar{I}_n, \bar{\beta}_{n-1}] \in \mathbb{R}^{4n-1}$ and the function, ie,

$$F_n(\xi_n) = f_n(\bar{x}_n) + \frac{1}{\delta_n} z_n\left[H_{n-1}(\xi_n) + l^*|g_n(\bar{x}_n)|\right] \tag{84}$$

and consider an approximation of the nonlinearity $F_n(\xi_n)$ within some compact set $\Omega_n \subset \mathbb{R}^{4n-1}$ by an LPA such that

$$F_n(\xi_n) := \theta_n^{*T}\Phi_n(\xi_n) + \varepsilon_{an}(\xi_n), \quad \forall \xi_n \in \Omega_n, \tag{85}$$

with $\theta_n^* \in \mathbb{R}^{\ell_n}$ being the optimal approximation weight, $\Phi_n(\xi_n) \in \mathbb{R}^{\ell_n}$ being the regressor vector, and $\varepsilon_{an}(\xi_n)$ being the approximation error. The error is bounded within Ω_n, ie, there exists some $\varepsilon_{Mn} > 0$ such that $|\varepsilon_{an}(\xi_n)| \le \varepsilon_{Mn}$ for all $\xi_n \in \Omega_n$. If we introduce an online estimation $\beta_n(t)$ of the optimal weight norm $\|\theta_n^*\|$ and functions $L_n := (1/2\gamma_n)\beta_n^2(t)$ and

$$P_n := V_n + L_n + \lambda_n\int_0^t V_n(s)ds, \tag{86}$$

then from (83)-(85), we have

$$\dot{P}_n \le \lfloor z_n\rceil_{\delta_n}\left[\theta_n^{*T}\Phi_n(\xi_n) + \varepsilon_{an}(\xi_n) + \frac{\lambda_n}{2}\lfloor z_n\rceil_{\delta_n} + g_n(\bar{x}_n)m(t)\alpha_n + d(t)\right] + \frac{1}{\gamma_n}\beta_n\dot{\beta}_n. \tag{87}$$

Using the Cauchy-Schwarz inequality, (87) yields

$$\dot{P}_n \le \lfloor z_n\rceil_{\delta_n}\left[\varepsilon_{an}(\xi_n) + \frac{\lambda_n}{2}\lfloor z_n\rceil_{\delta_n} + g_n(\bar{x}_n)m(t)\alpha_n + d(t)\right] + \|\theta_n^*\|\,\|\Phi_n(\xi_n)\|\,|\lfloor z_n\rceil_{\delta_n}| + \frac{1}{\gamma_n}\beta_n\dot{\beta}_n. \tag{88}$$

Define also the functions $Q_n := (1/2\gamma_n)(\tilde{\beta}_n^2 - \beta_n^2)$, where $\tilde{\beta}_n = \beta_n - \|\theta_n^*\|$ is the estimation error and

$$\overline{P}_n := P_n + Q_n. \tag{89}$$

If we select the estimation update law,

$$\dot{\beta}_n = \gamma_n \|\Phi_n(\xi_n)\| |\lfloor z_n \rceil_{\delta_n}|, \quad \beta_n(0) = 0, \tag{90}$$

then from Assumptions 2 and 3, (88)-(90), and (3), we obtain

$$\frac{d\overline{P}_n}{dt} \leq \lfloor z_n \rceil_{\delta_n} \left[\frac{\lambda_n}{2} \lfloor z_n \rceil_{\delta_n} + (\varepsilon_{Mn} + d^*) \frac{z_n}{\delta_n} + g_n(\overline{x}_n) m(t) \alpha_n \right] + \beta_n \|\Phi_n(\xi_n)\| |\lfloor z_n \rceil_{\delta_n}|. \tag{91}$$

Completing the square and using (3), we have that

$$\beta_n \|\Phi_n(\xi_n)\| |\lfloor z_n \rceil_{\delta_n}| \leq \frac{1}{2\delta_n} \left(\beta_n^2 + \|\Phi_n(\xi_n)\|^2 \right) \lfloor z_n \rceil_{\delta_n} z_n. \tag{92}$$

Thus, if we choose the control law,

$$u = \alpha_n = \kappa_n(P_n) \left(1 + |\beta_n|^2 + \|\Phi_n(\xi_n)\|^2 \right) z_n, \tag{93}$$

then from (4), (92), Equation (91) yields

$$\frac{d\overline{P}_n}{dt} \leq \left[\eta_n^* + g_n(\overline{x}_n) m(t) \kappa_n(P_n) \right] \Theta_n(t) \tag{94}$$

with

$$\Theta_n(t) := \left(1 + |\beta_n|^2 + \|\Phi_n(\xi_n)\|^2 \right) z_n \lfloor z_n \rceil_{\delta_n} \tag{95}$$

$$\eta_n^* := \max \left\{ \frac{\lambda_n}{2} + \frac{\varepsilon_{Mn} + d^*}{\delta_n}, \frac{1}{2\delta_n} \right\}. \tag{96}$$

It also holds true from the definition of Q_n and (90), (3), (95) that

$$\begin{aligned} |Q_n(t_2) - Q_n(t_1)| &= \frac{1}{\gamma_n} \|\theta_n^*\| \ |\beta_n(t_2) - \beta_n(t_1)| \\ &= \frac{1}{\gamma_n} \|\theta_n^*\| \left| \int_{t_1}^{t_2} \dot{\beta}_n(s) ds \right| = \int_{t_1}^{t_2} \|\theta_n^*\| \ \|\Phi_n(\xi_n)\| |\lfloor z_n \rceil_{\delta_n}| ds \\ &\leq \frac{1}{2\delta_n} \int_{t_1}^{t_2} \left(\|\theta_n^*\|^2 + \|\Phi_n(\xi_n)\|^2 \right) z_n \lfloor z_n \rceil_{\delta_n} ds \\ &\leq \frac{\max\{1, \|\theta_n^*\|^2\}}{2\delta_n} \int_{t_1}^{t_2} \Theta_n(s) ds \quad \forall t_2 \geq t_1. \end{aligned} \tag{97}$$

From (89), (94), (97), conditions (10)-(12) of Lemma 1 hold true, and therefore, P_n is upper bounded.

5 | BOUNDEDNESS AND TRACKING

Due to the integral terms in the virtual controls and the estimations β_i of the weight norms $\|\theta_i^*\|$, the closed-loop system state vector is of order $3n$. Hence, if we define the augmented state vector $x_{ag} = [x^T, I^T, \beta^T]^T$, then the dynamics of the closed-loop system can be written in the form $\dot{x}_{ag} = \overline{f}(x_{ag}, t)$ with $\overline{f} : R^{3n} \times [0, \infty) \to R^{3n}$ being a continuous with respect to x_{ag} and t vector field. Detailed calculations can verify the locally Lipschitz property with respect to x_{ag} of the vector field $\overline{f}$ in a neighborhood of $[x_0^T \quad 0]^T$. Thus, according to theorem 3.1 in the work of Khalil,[52] a unique solution exists within some time interval $[0, t_f)$. From the analysis in Section 4, we have proved that (9)-(12) hold true for all $P_i, \overline{P}_i, \Theta_i$ $(1 \leq i \leq n)$ defined in (64), (66), (73). If we also choose $\varepsilon_i < 1$ and apply Lemma 1, it is proved that P_i, V_i, L_i, $\int_0^t V_i(s)ds$ are bounded in $[0, t_f) \forall i = 1, \ldots, n$. The boundedness of $V_i, L_i, \int_0^t V_i(s)ds$ ensures that z_i, β_i, I_i are respectively bounded. Then, from $x_1 = z_1 + y_d$, the boundedness of x_1 is deduced. Inductively, we can prove that all x_i are bounded. Hence, assuming $x_1, \ldots, x_i$ are bounded, we will also prove that x_{i+1} is bounded. This follows directly since, from the boundedness of $\bar{x}_i$ assumption, $\alpha_i = A_i(\bar{x}_i, \bar{z}_i, \overline{I_{i+1}}, \bar{\beta}_i)$ is bounded, which, in turn, yields the boundedness of $x_{i+1} = z_{i+1} + \alpha_i$. Thus, the augmented state vector x_{ag} remains bounded and the final time t_f can be extended to infinity, ie, $t_f = +\infty$ (no finite explosion time).We have proved therefore that $\lceil z_i \rfloor_{\delta_i} \in \mathcal{L}_\infty \cap \mathcal{L}_2$ and $x_i, I_i, \beta_i, \alpha_i, u \in \mathcal{L}_\infty$. Then, from (47), we conclude that $\dot{z}_i \in \mathcal{L}_\infty$. Applying now the Barbalat's lemma, we have that $\lim_{t\to\infty} \lceil z_i(t) \rfloor_{\delta_i} = 0$, which, in turn, implies $\lim_{t\to\infty} |z_i(t)| \leq \delta_i$ and, finally, $\lim_{t\to\infty} |y(t) - y_d(t)| = \lim_{t\to\infty} |z_1(t)| \leq \delta_1$.

We note that the boundedness results of the proposed methodology are semiglobal (which is the standard norm for adaptive LPA control schemes) in the sense that the LPA approximation regions Ω_i should be chosen sufficiently large to include the region $\overline{\Omega}_i := \{\xi_i | \|\xi_i\| \leq C_{\xi,i}\}$ wherein the trajectory of $\xi_i(t)$ remains. Previous analysis has shown that there exist some constants $p, C_{x,i}, C_{z,i} > 0$ such that $\sum_{i=1}^{n} P_i(t) \leq p$, $\|\bar{x}_i(t)\| \leq C_{x,i}$, $\|\bar{z}_i(t)\| \leq C_{z,i}$ for all $t \geq 0$. Since

$$\sum_{i=1}^{n} \frac{1}{2\gamma_i} \beta_i(t)^2 + \sum_{i=1}^{n} \frac{(1-\varepsilon_i)\lambda_i}{2} I_i(t) \leq \sum_{i=1}^{n} P_i(t) \leq p, \tag{98}$$

we have that $\beta_i(t) \leq \sqrt{2\gamma_i p} := C_{\beta,i}$, $I_i(t) \leq 2p/[(1-\varepsilon_i)\lambda_i] := C_{I,i}$ for all $t \geq 0$, and therefore, $\|\xi(t)\| \leq \sqrt{C_{x,i}^2 + C_{z,i}^2 + C_{\beta,i}^2 + C_{I,i}^2} := C_{\xi,i}$ for all $t \geq 0$. Thus, for $\Omega_i := \{\xi_i \in \mathbb{R}^{4i-1} | \|\xi_i\| \leq \overline{C}_{\xi,i}\}$, one must select $\overline{C}_{\xi,i} \geq C_{\xi,i}$ for the results to be valid.

Thus, we have now proved the following theorem, which is the main result of this paper showing that the backstepping design described in the previous section ensures approximate output tracking and boundedness of all the closed-loop signals.

Theorem 1. *Consider the strict-feedback nonlinear system described by (5) and a reference trajectory y_d satisfying Assumptions 1 to 4. If we select the control input according to Equation (93) where the virtual control laws are given by Equation (71) ($\varepsilon_i < 1$), the estimator update laws (67), and the LPA approximation regions $\Omega_i := \{\xi_i \in \mathbb{R}^{4i-1} | \|\xi_i\| \leq \overline{C}_{\xi,i}, \overline{C}_{\xi,i} \geq C_{\xi,i}\}$, then all the closed-loop signals are bounded and* $\lim_{t\to\infty} |y(t) - y_d(t)| \leq \delta_1$.

Remark 3. The proposed control scheme avoids the "explosion of complexity" problem that occurs in standard backstepping control since there is no need to calculate the time derivative of the virtual control laws or its partial derivatives with respect to the state variables.[46] This is achieved due to Lemma 2 where the bounds $|\dot{\alpha}_{i-1}| \leq H_{i-1}(\bar{x}_i, \bar{z}_i, \bar{I}_i, \bar{\beta}_{i-1})$ are obtained and the use of LPAs to estimate functions of those bounds (see (61)). This constitutes an alternative and more direct approach than dynamic surface control in which the virtual control laws typically pass through first-order filters.[46]

Remark 4. With respect to the controller parameter selection, the following observations can be made after running several simulations with different parameter values. Design parameter δ_1 is predetermined from the desired tracking accuracy of the problem under study. Parameters δ_i ($i \geq 2$), on the other hand, should be typically chosen bigger to avoid large input transients but not extremely big as this would increase significantly the convergence time. Similarly, increasing the adaptation gains γ_i and/or parameters λ_i appears to result in a faster convergence rate at the expense of larger input transients.

6 | SIMULATION STUDY

To verify our theoretical analysis, we consider the following second-order nonlinear system:

$$\begin{aligned} \dot{x}_1 &= x_1 \cos(x_1) + \left(1 + \sin(x_1)^2\right) x_2, \quad y = x_1 \\ \dot{x}_2 &= x_1 x_2 - \left(2\sqrt{x_1^2 + x_2^2} + 0.5\right) h(u,t) \end{aligned} \tag{99}$$

with a nonsymmetric deadzone nonlinearity as follows:

$$h(u,t) = \begin{cases} (2+\sin t)(1-0.3\sin u)(u-2.5), & \text{if } u > 2.5 \\ 0, & \text{if } -1.5 \leq u \leq 2.5 \\ (2+\sin t)(0.8-0.2\cos u)(u+1.5), & \text{if } u < -1.5. \end{cases}$$

Our objective is for the output y to follow the reference signal $y_d(t) = \sin t$. The applied control to the system is designed according to the procedure described in the previous section. Initially, the virtual control a_1 is calculated as

$$\alpha_1 = \kappa_1 \left[\frac{1}{2} \lfloor z_1 \rceil_{\delta_1}^2 + \frac{1}{2\gamma_1}\beta_1^2 - \frac{\varepsilon_2\lambda_2}{2} \int_0^t \lfloor x_2(s) - \alpha_1(s) \rceil_{\delta_2}^2 ds + \frac{\lambda_1}{2} \int_0^t \lfloor z_1(s) \rceil_{\delta_1}^2 ds \right] \left(1 + \|\Phi_1(x_1, z_1)\|^2 + \beta_1^2\right) z_1 \tag{100}$$

with $z_1 = x_1 - y_d$ and β_1 update law,

$$\dot{\beta}_1 = \gamma_1 \|\Phi_1(x_1, z_1)\| |\lfloor z_1 \rceil_{\delta_1}|.$$

Then, the applied control input u is given as

$$u = \kappa_2 \left[\frac{1}{2} \lfloor z_2 \rceil_{\delta_2}^2 + \frac{1}{2\gamma_2}\beta_2^2 + \frac{\lambda_2}{2} \int_0^t \lfloor z_2(s) \rceil_{\delta_2}^2 ds \right] \left(1 + \|\Phi_2(x_1, x_2, z_1, z_2, I_1, I_2, \beta_1)\|^2 + \beta_2^2\right) z_2,$$

with $z_2 = x_2 - \alpha_1$, $I_i = \int_0^t \lfloor z_i(s) \rceil_{\delta_i}^2 ds$ $(i = 1, 2)$, and β_2 update law,

$$\dot{\beta}_2 = \gamma_2 \|\Phi_2(x_1, x_2, z_1, z_2, I_1, I_2, \beta_1)\| |\lfloor z_2 \rceil_{\delta_2}|.$$

In our simulation scenario, we considered initial conditions $x_1(0) = x_2(0) = 1$ and selected controller parameters $\delta_1 = 0.06$, $\delta_2 = 1$, $\lambda_1 = \lambda_2 = 0.1$, $\gamma_1 = \gamma_2 = 1/2$, $\varepsilon_2 = 0.25$ and functions $\kappa_1(x) = \kappa_2(x) = x^2 \cos x$. Two second-order neural networks with 6 and 36 neurons, respectively, were used to implement the regressor vectors $\Phi_1(x_1, z_1)$, $\Phi_2(\xi_2)$ with activation functions $\tanh(\cdot)$. These include terms that are products up to second order of the activated network inputs, eg, $\Phi_1(x_1, z_1) = [1, \tanh(x_1), \tanh(z_1), \tanh^2(x_1), \tanh^2(z_1), \tanh(x_1)\tanh(z_1)]^T$.

For comparison, we have included simulations of the Nussbaum controller described in eqs. (67)-(69) in the work of Zhang et al[46] with parameters $k_1 = 3$, $k_2 = 20$, $a_1 = 1$, $a_2 = 0.1$, $\gamma_1 = 1$, $\gamma_2 = 5$, and $\sigma_1 = \sigma_2 = 10^{-3}$.

Simulation results for the proposed adaptive nonlinear PI and the Nussbaum controller in the work of Zhang et al[46] are shown in Figures 1 to 3. As expected, for both controllers, the output y approximately tracks the reference signal y_d (see Figure 1) with bounded control inputs u (see Figure 2). For the particular selection of control parameters, the Nussbaum controller yields faster convergence at the expense of higher input transients. In addition, both parameter estimates β_1 and β_2 remain bounded (see Figure 3).

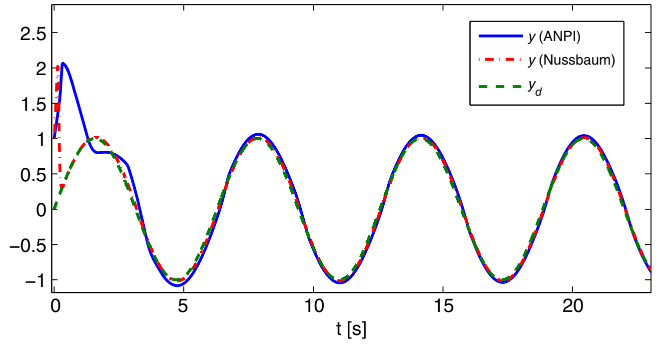

FIGURE 1 Time responses of the output signal y for the adaptive nonlinear PI (ANPI) controller and the Nussbaum controller in the work of Zhang et al[46] [Colour figure can be viewed at wileyonlinelibrary.com]

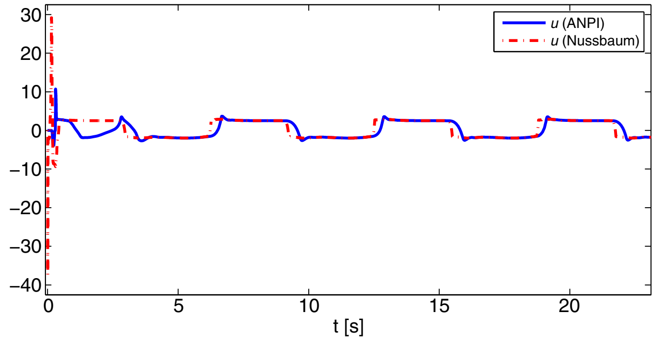

FIGURE 2 Time response of the control input u for the adaptive nonlinear PI (ANPI) controller and the Nussbaum controller in the work of Zhang et al[46] [Colour figure can be viewed at wileyonlinelibrary.com]

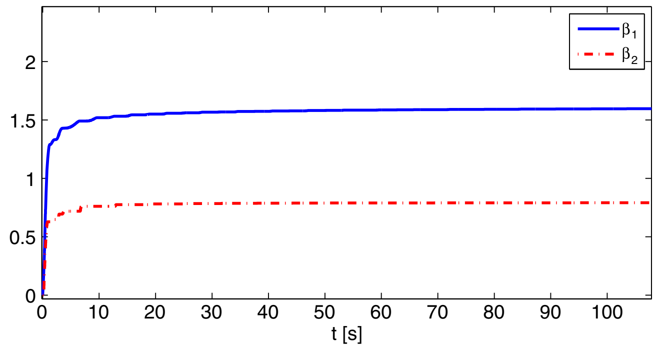

FIGURE 3 Time response of the adaptation variables β_1 and β_2 [Colour figure can be viewed at wileyonlinelibrary.com]

We note that implementing the controller in the work of Zhang et al[46] involves several extra calculations since the partial derivatives $\frac{\partial\alpha_1}{\partial x_1}$, $\frac{\partial\alpha_1}{\partial x_d}$, $\frac{\partial\alpha_1}{\partial\zeta_1}$, and $\frac{\partial\alpha_1}{\partial\hat{\theta}_1}$ are needed in this case. For larger system dimensions, obtaining those partial derivatives $\frac{\partial\alpha_i}{\partial x_j}$ $(1 \leq j \leq i)$ becomes increasingly tedious making the implementation extremely difficult. As explained in Remark 3, this problem is avoided with our design.

7 | CONCLUSIONS

For the first time, an adaptive control extension of the nonlinear PI methodology has been developed in this work. Specifically, we proposed an adaptive nonlinear PI backstepping LPA controller for strict-feedback SISO nonlinear systems with nonsmooth actuator nonlinearities. New theoretical results have been obtained that made possible the analysis of the combined backstepping nonlinear PI method with the adaptive control technique. Using these results, we proved that the proposed controller achieves approximate output tracking and boundedness of all signals in the closed-loop. Future work may consider extensions to MIMO systems and output feedback control.

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